Eton 2021 King's Scholarship B Solutions

1) a) $900^2 - 1 = 29 \times 31 = 899$

b) $x^3 + 1$

c) i) $27001 = 27000 + 1 = 30^3 + 1 = 31 \times (900 - 30 + 1)$

ii) $x^2 - x + 1 = 211$

$x^2 - x - 210 = 0$

$x = 15$

$15^3 + 1 = 3376$

iii) $2^{48} + 1 = (2^{16} + 1)(2^{32} - 2^{16} + 1)$

$= 65537(2^{32} - 2^{16} + 1)$

iv) $5^{18} + 1 = (5^6 + 1)(5^{12} - 5^6 + 1)$

$= 15626(5^{12} - 5^6 + 1)$

$\frac{15626}{13} = 1202$

2) a) £150

b) $80a + 130b + \frac{2}{3} \times 30(a + b)$

$= 100a + 150b = 800$

$2a + 3b = 16$

a,b= 8,0; 5,2; 2,4 but a+b is a multiple of 3 so 2,4, which gives 6.

3) a) $\left(4 + 3\sqrt{5}\right)\left(4 + 3\sqrt{5}\right) = 16 + 45 + 24\sqrt{5} = 61 + 24\sqrt{\ }$

b) $11 - 4\sqrt{7}$

c) i) $1 + \sqrt{6}$

ii) $\sqrt{5} - 2$

iii) $\sqrt{2} - 1$

d) $\frac{1+\sqrt{7}}{\sqrt{2}}$

4)

	Walrus	Carpenter	Total
Single Lion pail	2m	3m	5m
Single Unicorn pail	5n	4n	9n

We also know that $5m : 9n = 3 : 5.$

So $\frac{5m}{3} = \frac{9n}{5}$

and $25m = 27n$

and $m : n = 27 : 25.$

Let m=27x and n=25x.

Substituting back into our ratios:

	Walrus	Carpenter	Total
Single Lion pail	54x	81x	135x
Single Unicorn pail	125x	100x	225x

Now adding the number of oysters eaten from l lion pails and u unicorn pails, and since the Walrus and Carpenter eat the same number of oysters:

$54xl + 125xu = 81xl + 100xu$

$54l + 125u = 81l + 100u$

$25u = 27l$

So the lowest values of l and u are 25 and 27.

5) a) i) $2^3 = 8$

ii) $2^4 = 16$

iii) 2^n

b) i) 2

ii) 3

iii) 5

c) i) 34

ii) A string with n letters can be viewed as a string of n-2 preceded by two letters, which can be:

- oo… in which case the word is impolite so contributes zero polite words to our total
- gg… in which case the politeness is solely determined by the n-2 letters following so contributes O_{n-2} polite words to our total
- go… or og… in which case, because the first two letters are opposite to each other the politeness is determined by the last n-1 letters of the word (thinking of the two combinations collectively). The first two letters can never determine politeness as they are opposite so go… is equivalent to o… and og… is equivalent to o… Thus, these two options together contribute O_{n-1} polite words to our total.

So, we get $O_n = O_{n-1} + O_{n-2}$

6) a) For one lap:

	Tweedle-Dum (TU)	Tweedle-Dee (TE)
Distance	d	d
Speed	7	8
Time	$\frac{d}{7}$	$\frac{d}{8}$

Total time taken is the same so if Tweedle-Dum does u laps and Tweedle-Dee does e laps then the time taken by each is $\frac{ud}{7} = \frac{ed}{8}$. So 8u=7e and 8 is a factor of e and 7 is a factor of u, if they are ever going to overlap at the start line again (which according to the question they do). So the smallest number is 7 laps for Tweedle-Dum and 8 laps for Tweedle-Dee.

The total distance covered by Tweedle-Dum is 7d.

Thinking of the distance between each meeting:

- the distance covered by Tweedle-Dum is 7x (where x is the time between meetings); and
- the distance covered by Tweedle-Dee is 8x; and
- so 7x+8x=d
- so $x = \frac{d}{15}$
- so the distance covered by Tweedle-Dum between meetings is $\frac{7d}{15}$.

Tweedle-Dum covers 7d in total and this journey is divided into $\frac{7d}{\frac{7d}{15}} = 15$ sections.

15 sections means 14 dividers between sections, i.e. 14 meetings (or 16 including start and finish).

b)

	Wind against	Still weather	Wind with
Distance	**s-x**	**st**	**0.8t(s+x)**
Speed	s-x	s	s+x
Time	1	t	0.8t

st=0.8t(s+x) gives s=4x. Table then becomes:

	Wind against	Still weather	Wind with
Distance	**3x**	**4xt**	**4xt**
Speed	3x	4x	5x
Time	1	t	0.8t

So 3x=4xt and t=3/4. So 0.8t=3/5.

Total journey time = $1\frac{3}{5} hours = 1\ hour\ 36\ minutes$

7) a) Special triangle with $90°, 60°\ and\ 30°$.

Ratio of sides is as usual 1 : $\sqrt{3}$: 2.

In this triangle sides are therefore 3 : $3\sqrt{3}$: 6.

Area = $\frac{1}{2} \times b \times h = \frac{1}{2} \times 6 \times 3\sqrt{3} = 9\sqrt{3}$

b) Each of the grey triangles is half an equilateral triangle.

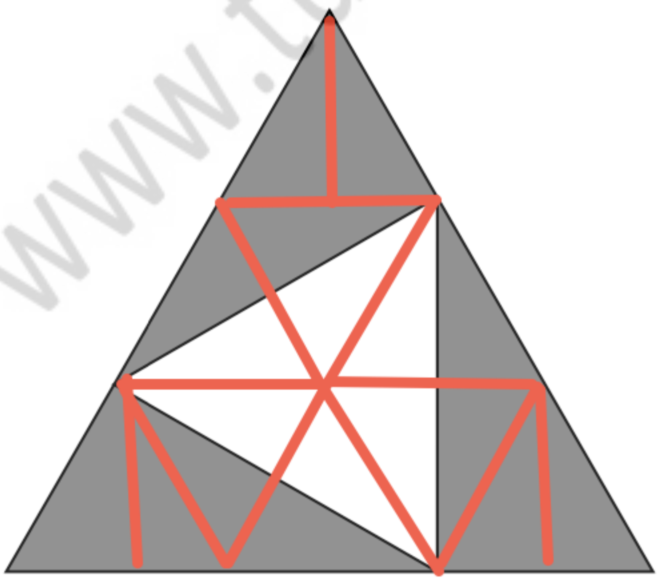
Make the shortest length a then the height is $\sqrt{3}a$ and the hypotenuse $2a$ and the area of the three grey triangles is $3 \times \frac{a \times \sqrt{3}a}{2} = \frac{3\sqrt{3}a^2}{2}$

The area of the entire large triangle is $\frac{1}{2} \times 3a \times \left(\sqrt{3} \times \frac{3a}{2}\right) = \frac{9\sqrt{3}a^2}{4}$

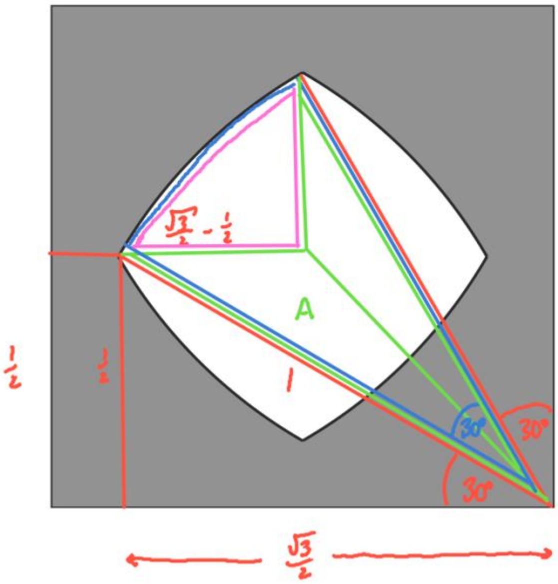
So the area of the white triangle is $\frac{3\sqrt{3}a^2}{4}$.

Therefore the ratio is 2:1.

Or: each of the triangles in the following diagram is congruent and by counting the ratio is 2:1.



c)



Let side length be 1 (since a ratio question)

Sector area $= \frac{30}{360} \times \pi \times 1^2 = \frac{\pi}{12} = x$

Δ A area $= \frac{1}{2}bh = \frac{1}{2}\left(\frac{\sqrt{3}-1}{2}\right) \times \frac{1}{2} = y$

Pink area $= x - 2y = \frac{\pi}{12} - \frac{\sqrt{3}-1}{4}$

White area $= 4 \times$ pink area $= \frac{\pi}{3} - \sqrt{3} + 1$

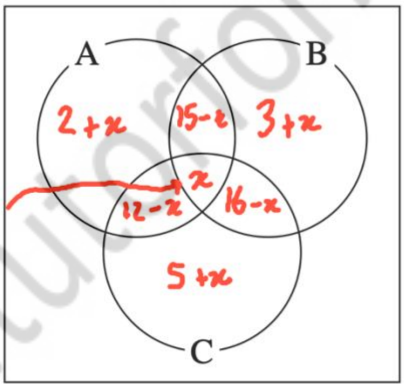
Grey area $= 1 - \left(\frac{\pi}{3} - \sqrt{3} + 1\right)$

Ratio grey : white $= 0.6849 : 0.3151$

$= 1 : 0.460$

8) a)

Start here



Sum = 57

(A) + 3+x + 16-x + 5+x = 57

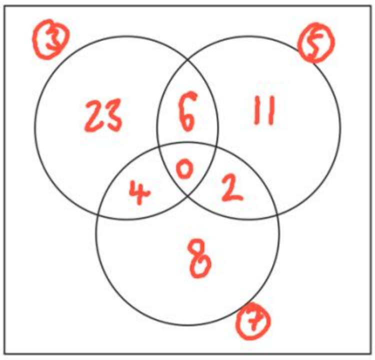
2A

53 + x = 57

x = 4

b)

Start at centre



3,5,7	105	→ 0
3,5	15	→ 6
3,7	21	→ 4
5,7	35	→ 2
3	33	
5	19	
7	14	

(54)

c) Number of ways in which 7 students can enrol in 3 classes without limitation on numbers of students in each class = 3^7

However, quite a lot of those options have students in exactly two classes:

- number of ways that all the students can do acrobatics and ballet is 2^7 but then subtract two, which are all students doing acrobatics or all students doing ballet
- similarly for acrobatics and capoeira, and ballet and capoeira
- so the number of ways the students can do exactly two classes = 3×2^7 - 6

Then there's also the possibility that they are all enrolled in the same class, which is 3 options.

So the overall answer is $3^7 - (3 \times 2^7 - 6) - 3 = 1803$ ways.