Eton College King's Scholarship Examination 2021

MATHEMATICS B

(**One and a half hours**)

Candidate number: _____

Please write your candidate number on EVERY sheet.

Please answer on the paper in the spaces provided.

There are 8 questions: each one is worth 10 marks.
Calculators are allowed, but you should show all your working.

Do not turn over until told to do so.

1. (a) By considering $(x + 1)(x - 1) = x^2 - 1$, find two prime factors of 899.

(b) Multiply out and simplify fully $(x + 1)(x^2 - x + 1)$.

(c) Using this result, prove that

(i) 27001 is divisible by 31

(ii) 3376 is divisible by 211

(iii) $2^{48} + 1$ is divisible by 65537

(iv) $5^{18} + 1$ is divisible by 13

2. (a) I am buying musical instruments. A ukulele and a xylophone together cost £120 more than a violin. A violin and a xylophone together cost £180 more than a ukulele. How much does a xylophone cost?

(b) Some of the instruments are presents for my godchildren. Because I dislike their parents, I have decided to buy each one either an accordion or a set of bagpipes. Accordions cost £80 and bagpipes cost £130. Two thirds of the presents will need to be sent by post: postage costs £30 per instrument. I spend £800 altogether. How many godchildren do I have? [Your working should demonstrate that your solution is unique.]

[space for continuation of solution to 2(b), if required]

3. (a) Show carefully by multiplying out and simplifying that $(4 + 3\sqrt{5})^2 = 61 + 24\sqrt{5}$.

(b) Multiply out and simplify $(\sqrt{7} - 2)^2$.

(c) Use the pattern you have observed to find an expression for the following in the form $a + b\sqrt{c}$, where a, b and c are integers.

(i) $\sqrt{7 + 2\sqrt{6}}$

(ii) $\sqrt{9 - 4\sqrt{5}}$

(iii) $\sqrt{3-2\sqrt{2}}$

(d) Find an expression for $\sqrt{4+\sqrt{7}}$ in the form $\frac{1+\sqrt{q}}{\sqrt{p}}$, where p and q are integers.

4. Oysters are sold in pails by two rival companies, *Lion* and *Unicorn*. The Walrus and the Carpenter always divide up the contents of a *Lion* brand pail between themselves in the ratio 2:3, while they divide up a *Unicorn* brand pail in the ratio 5:4. The ratio of the total number of oysters in a *Lion* brand pail to that in a *Unicorn* brand pail is 3:5. If they cannot resist polishing off all the oysters in any pail they start, how many pails of each brand do they eat given that they consume the smallest total number of pails they need in order to ensure that they eat the same number of oysters each?

[space for continuation of solution to 4, if required]

5. Oggish is the language of the land of Og. Its alphabet consists of just two letters, ‘o’ and ‘g’. Words may be formed by any combination of letters, but any word which contains the sequence of letters ‘oo’ is considered obscene and is not used in polite Oggish society. Thus, for example, the words ‘oggg’, ‘gogogogo’ and ‘ggggggg’ are considered polite, while ‘oogo’, ‘ggggoooo’ and ‘googoog’ are obscene.

   (a) How many words in total (i.e. both polite and obscene) are there which contain

      (i) exactly 3 letters?

      (ii) exactly 4 letters?

      (iii) exactly n letters?

   (b) How many polite words are there which have

      (i) exactly one letter?

(ii) exactly two letters?

(iii) exactly three letters?

(c) Oggo the Grammarian claims that if O_n is the number of polite Oggish words of n letters, then

$$O_n = O_{n-1} + O_{n-2}$$

(i) Use this formula to find the number of polite Oggish words which have exactly seven letters.

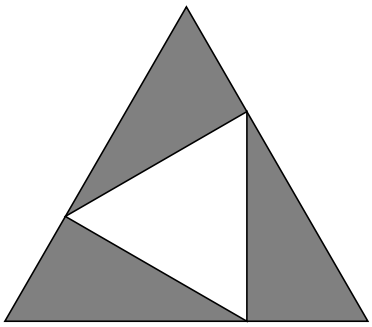
(ii) Explain Oggo's reasoning carefully.

6. (a) Tweedle-Dum and Tweedle-Dee decide to have a running race on a circular track. They start off in opposite directions from the finish line and stop when first they both meet there again. If Tweedle-Dum runs at a steady 7mph and Tweedle-Dee at a steady 8mph, how many times will they pass each other in between?

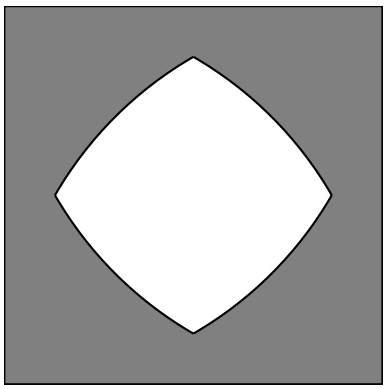
(b) The Gryphon flies against the wind from the Duchess's House to the Queen's Croquet Ground in an hour and then flies back again with the wind in 80% of the time it would have taken him in still air. How long did the entire round trip take?

7. (a) Use Pythagoras's theorem to show that an equilateral triangle of perimeter 18 has an area of $9\sqrt{3}$.

(b) The design for a stained glass window is shown below. The whole window is in the shape of an equilateral triangle. Three identical right-angled triangles (of coloured glass) just touch inside it, and enclose a triangle of plain glass. What is the ratio of coloured to plain glass in the whole window?

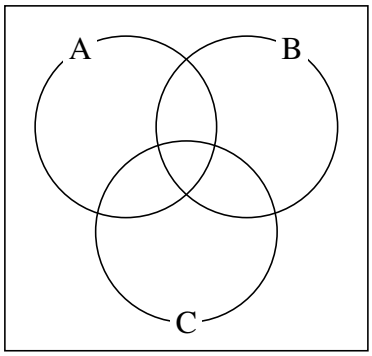


(c) Another window has a design which is shown below. The whole window is a square; the central section is made of plain glass; the outer section is coloured. The curved lines are arcs of circles of the same radius as the side length of the square and centred on its corners. What is the ratio of coloured to plain glass in the whole window? Give your answer in the form $1 : x$, where x is a decimal correct to 3 sf.

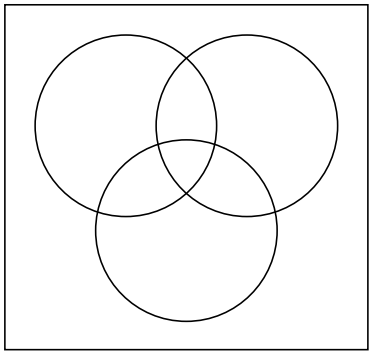


[space for continuation of solution to 7(c), if required]

8. (a) 57 people speak at least one of Aragonese, Basque and Castilian. 29 speak Aragonese; 34 speak Basque; 33 speak Castilian; 15 speak both Aragonese and Basque; 16 speak both Basque and Castilian; 12 speak both Aragonese and Castilian. How many speak all three languages? You may find it helpful to use the diagram in your answer.



(b) Of the positive integers strictly less than 100, how many are divisible by at least one of 3, 5 and 7? You may find it helpful to use the diagram in your answer.



(c) Groucho, Chico, Harpo, Gummo, Zeppo, Karl and Spencer must each choose to enrol in exactly one of three classes: Acrobatics, Ballet or Capoeira. In how many ways can this happen if there must be at least one student in each class? [Note that the actual participants in each class matter, not just the numbers in each class, i.e. "Groucho, Chico, Harpo, Gummo and Zeppo all choose Acrobatics, while Karl chooses Ballet and Spencer chooses Capoeira" is a different result from "Chico, Harpo, Gummo, Zeppo and Spencer all choose Acrobatics, while Karl chooses Ballet and Groucho chooses Capoeira".]

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